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On complementary channels and the additivity problem

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Abstract

We explore complementarity between output and environment of a quantum channel (or, more generally, CP map), making an observation that the output purity characteristics for complementary CP maps coincide. Hence, validity of the mutiplicativity/additivity conjecture for a class of CP maps implies its validity for complementary maps. The class of CP maps complementary to entanglement-breaking ones is described and is shown to contain diagonal CP maps as a proper subclass, resulting in new class of CP maps (channels) for which the multiplicativity/additivity holds. Covariant and Gaussian channels are discussed briefly in this context.

In what follows $\mathcal{H}_A, \mathcal{H}_B, \dots$ will denote (finite dimensional) Hilbert spaces of quantum systems $A, B, \dots \mathfrak{M}(\mathcal{H})$ denotes the algebra of all operators, $\mathfrak{S}(\mathcal{H})$–– the convex set of density operators (states) and $\mathfrak{P}(\mathcal{H}) = \mathrm{ext}\mathfrak{S}(\mathcal{H})$ – the set of pure states (one-dimensional projections) in $\mathcal{H}$. For a natural d, $\mathcal{H}_d$ denotes the Hilbert space of $d-$dimensional complex vectors, and $\mathfrak{M}_d$ – the algebra of all complex $d \times d-$ matrices.

Given three finite spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ and a linear operator $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$, the relation

$$\Phi_B(\rho) = \mathrm{Tr}_{\mathcal{H}_C} V\rho V^*, \quad \Phi_C(\rho) = \mathrm{Tr}_{\mathcal{H}_B} V\rho V^*; \quad \rho \in \mathfrak{M}(\mathcal{H}_A) \tag{1}$$

defines two CP maps $\Phi_B : \mathfrak{M}(\mathcal{H}_A) \to \mathfrak{M}(\mathcal{H}_B)\,,\ \Phi_C : \mathfrak{M}(\mathcal{H}_A) \to \mathfrak{M}(\mathcal{H}_C)\,,$

which will be called mutually *complementary*. If V is an isometry, both maps are trace preserving (TP) i.e. channels. The name "complementary

channels" is taken from the paper [4], where they were used to define quantum version of degradable channels.

The Stinespring dilation theorem implies that for given a CP map (channel) a complementary always exists. In the Appendix we give a proof which also clarifies in what sense the complementary map is unique. It follows that for a given CP map Φ_B, any two channels $\Phi_C, \Phi_{C'}$ complementary to Φ_B are equivalent in the sense that there is a partial isometry $W : \mathcal{H}_C \to \mathcal{H}_{C'}$ such that

$$\Phi_{C'}(\rho) = W\Phi_C(\rho)W^*, \quad \Phi_C(\rho) = W^*\Phi_{C'}(\rho)W, \tag{2}$$

for all ρ. Dilations with the minimal dimensionality d_C are called *minimal*. Any two minimal dilations are isometric (i.e. W is an isometry from $\mathcal{H}_C$ onto $\mathcal{H}_{C'}$). By performing a Stinespring dilation for a complementary CP map one obtains a map equivalent to the initial one in the sense (2). Thus the complementarity is a relation between the equivalence classes of CP maps.

To simplify formulas we shall also use the notation $\tilde{\Phi}$ for the map which is complementary to Φ.

Consider the following "measures of output purity" of a CP map Φ

$$\nu_p(\Phi) = \max_{\rho \in \mathfrak{S}(\mathcal{H})} [\mathrm{Tr}\Phi(\rho)^p]^{1/p}, \quad 1 \leq p, \tag{3}$$

introduced in [1]. For $p = \infty$ one puts $\nu_\infty(\Phi) = \max_{\rho\in\mathfrak{S}(\mathcal{H})} \|\Phi(\rho)\|$. In the case of channel Φ, further useful characteristics are the minimal output entropy

$$\check{H}(\Phi) = \min_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho)),$$

where $H(\sigma) = -\mathrm{Tr}\sigma \ln \sigma$ is the von Neumann entropy of a density operator σ, and its *convex closure*

$$\hat{H}_\Phi(\rho) = \min_{\rho = \sum_x \pi(x)\rho(x)} \sum_x \pi(x) H(\Phi(\rho(x))),$$

where the minimum is taken over all possible convex decompositions of the density operator ρ into pure states $\rho(x) \in \mathfrak{S}(\mathcal{H})$ [8]. By convexity argument, all these quantities remain unchanged if we replace $\mathfrak{S}(\mathcal{H})$ by $\mathfrak{P}(\mathcal{H})$ in their definitions.

Theorem 1 *If one of the relations*

$$\nu_p(\Phi_1 \otimes \Phi_2) = \nu_p(\Phi_1)\, \nu_p(\Phi_2), \tag{4}$$

$$\check{H}\left(\Phi_1 \otimes \Phi_2\right) = \check{H}\left(\Phi_1\right) + \check{H}\left(\Phi_2\right), \tag{5}$$

$$\hat{H}_{\Phi_1 \otimes \Phi_2}(\rho_{12}) \geq \hat{H}_{\Phi_1}(\rho_1) + \hat{H}_{\Phi_2}(\rho_2) \tag{6}$$

holds for the CP maps (channels) Φ_1, Φ_2, *then similar relation holds for the pair of their complementary maps* $\tilde{\Phi}_1, \tilde{\Phi}_2$. *If one of these relations holds for given* Φ_1 *and arbitrary* Φ_2, *then similar relation holds for complementary* $\tilde{\Phi}_1$ *and arbitrary* Φ_2.

Remark. Let us recall that for two given channels Φ_1, Φ_2, the property (4) with $p \in [1, 1+\varepsilon]$ implies (5) by differentiation [1]. The property (6), which is equivalent to the additivity of the $\chi-$capacity (the Holevo capacity) with arbitrary input constraints [8], implies both additivity of the $\chi-$capacity and (5) by the arguments similar to that for the superadditivity of entanglement of formation, see e. g. [17]. On the other hand, assuming that (4) with $p \in [1, 1+\varepsilon]$ holds for all CP maps Φ_1, Φ_2 implies (5), (6) for all channels, and these two properties, as well as additivity of the $\chi-$capacity, are globally equivalent, i. e. if one holds for all channels, another holds for all channels as well [17].

Proof. If $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}_A$, then Hermitian operators $\Phi(\rho), \tilde{\Phi}(\rho)$ have the same nonzero eigenvalues. Indeed, $\Phi(\rho), \tilde{\Phi}(\rho)$ are partial traces of the operator $|\psi_{BC}\rangle\langle\psi_{BC}|$, where $|\psi_{BC}\rangle = V|\psi\rangle \in \mathcal{H}_B \otimes \mathcal{H}_C$, then the proof goes in the same way as in the case of normalized vectors (see, e.g. [15], Theorem 2.7).

Both $\mathrm{Tr}\sigma^p$ and $H(\sigma)$ are universal functions of nonzero eigenvalues of a Hermitian operator σ. From the definitions of $\nu_p, \check{H}$ and $\hat{H}$ it follows that for arbitrary CP map Φ

$$\nu_p(\tilde{\Phi}) = \nu_p(\Phi). \tag{7}$$

Moreover, if Φ is a channel, then

$$\check{H}(\tilde{\Phi}) = \check{H}(\Phi), \tag{8}$$

$$\hat{H}(\tilde{\Phi}) = \hat{H}(\Phi). \tag{9}$$

Now notice that if $\Phi_j, \tilde{\Phi}_j, j = 1, 2$, are two pairs of complementary CP maps, then $\Phi_1 \otimes \Phi_2$ and $\tilde{\Phi}_1 \otimes \tilde{\Phi}_2$ are complementary. For this take $\mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}, \mathcal{H}_C = \mathcal{H}_{C_1} \otimes \mathcal{H}_{C_2}$ and $V = V_1 \otimes V_2$. Summarizing all these facts, we get the statement. □

Assume that a CP map $\Phi : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{H}')$ is given by Kraus representation

$$\Phi(\rho) = \sum_{\alpha=1}^{\tilde{d}} V_\alpha \rho V_\alpha^*, \tag{10}$$

then a complementary map $\tilde{\Phi} : \mathcal{M}(\mathcal{H}) \to \mathcal{M}_{\tilde{d}}$ is given by

$$\tilde{\Phi}(\rho) = \left[\mathrm{Tr} V_\alpha \rho V_\beta^*\right]_{\alpha,\beta=\overline{1,\tilde{d}}} = \left[\mathrm{Tr} \rho V_\beta^* V_\alpha\right]_{\alpha,\beta=\overline{1,\tilde{d}}}, \tag{11}$$

since $V = \sum_{\alpha=1}^{\tilde{d}} \oplus V_\alpha$ is a map from $\mathcal{H}$ to $\sum_{\alpha=1}^{\tilde{d}} \oplus \mathcal{H}' \simeq \mathcal{H}' \otimes \mathcal{H}_{\tilde{d}}$ for which $\Phi, \tilde{\Phi}$ are given by the partial traces (1), see [7]. By writing the trace in $\mathcal{H}'$ with respect to an orthonormal basis $\{e_j'\}$, we have the Kraus representation

$$\tilde{\Phi}(\rho) = \sum_{j=1}^{d'} \tilde{V}_j \rho \tilde{V}_j^*, \tag{12}$$

where $(\tilde{V}_j)_\alpha = \langle e_j' | V_\alpha$. One can check by direct computation that applying the same procedure to $\tilde{\Phi}$, one obtains the map $\tilde{\tilde{\Phi}}$ which is isometric to Φ.

A CP map $\Phi : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{H}')$ is *entanglement-breaking* if it has a Kraus representation with rank one operators V_α [10]:

$$\Phi(\rho) = \sum_{\alpha=1}^{\tilde{d}} |\varphi_\alpha\rangle\langle\psi_\alpha|\rho|\psi_\alpha\rangle\langle\varphi_\alpha|. \tag{13}$$

Such a CP map is channel if and only if the (over)completeness relation

$$\sum_{\alpha=1}^{\tilde{d}} |\psi_\alpha\rangle\langle\varphi_\alpha|\varphi_\alpha\rangle\langle\psi_\alpha| = I$$

is fulfilled. The complementary map $\tilde{\Phi} : \mathcal{M}(\mathcal{H}) \to \mathcal{M}_{\tilde{d}}$ is

$$\tilde{\Phi}(\rho) = \left[c_{\alpha\beta}\langle\psi_\alpha|\rho|\psi_\beta\rangle\right]_{\alpha,\beta=\overline{1,\tilde{d}}}, \tag{14}$$

where $c_{\alpha\beta} = \langle\varphi_\beta|\varphi_\alpha\rangle$. Notice that by the Kolmogorov decomposition, arbitrary nonnegative definite matrix can be represented in such form. In the special case where $\{\psi_\alpha\}_{\alpha=\overline{1,\tilde{d}}}$ is an orthonormal base in $\mathcal{H}$, (14) is *diagonal* CP map [11]. Diagonal channels, which are characterized by additional

property $c_{\alpha\alpha} \equiv 1$, were also earlier considered in [4] under the name of dephasing channels. From (13) we see that the diagonal maps are complementary to a particular class of entanglement-breaking maps, namely to c-q maps. For another special subclass of entanglement-breaking maps, the q-c maps, $\{\varphi_\alpha\}_{\alpha=\overline{1,\tilde{d}}}$ is an orthonormal base in $\mathcal{H}$, so that $c_{\alpha\beta} = \delta_{\alpha\beta}$, and the complementary map is easily seen to be of the same subclass.

Let us rewrite (14) in the form

$$\tilde{\Phi}(\rho) = \sum_{\alpha,\beta=1}^{\tilde{d}} c_{\alpha\beta} |e_\alpha\rangle\langle\psi_\alpha|\rho|\psi_\beta\rangle\langle e_\beta|$$

where $\{e_\alpha\}$ is the canonical base for $\mathcal{H}_{\tilde{d}}$. Representing $c_{\alpha\beta} = \sum_{j=1}^{d'} \bar{v}_{\beta j} v_{\alpha j}$ by Kolmogorov decomposition and denoting

$$\tilde{V}_j = \sum_{\alpha=1}^{\tilde{d}} v_{\alpha j} |e_\alpha\rangle\langle\psi_\alpha|, \tag{15}$$

we have the Kraus representation (12) for the complementary map. For the diagonal maps $|\psi_\alpha\rangle = |e_\alpha\rangle$, hence from (15) one sees that the diagonal maps are characterized by the property of having a Kraus representation with simultaneously diagonalizable (i.e. commuting normal) operators $\tilde{V}_j$. Somewhat more generally, $\{|\psi_\alpha\rangle\}$ can be an orthonormal base different from $\{|e_\alpha\rangle\}$, in which case both $\tilde{V}_k^*\tilde{V}_j$ and $\tilde{V}_j\tilde{V}_k^*$ are families of commuting normal operators.

For entanglement-breaking channels the additivity property (5) (and in fact, (6), although not explicitly stated) with arbitrary second channel was established by Shor [16] and the multiplicativity property (4) for all $p > 1$ by King [12], using the Lieb-Thirring inequality. This proof of multiplicativity can be generalized with almost no changes to the case of entanglement-breaking CP maps. Note that for diagonal channels (expression (14) with $\{|\psi_\alpha\rangle\} = \{|e_\alpha\rangle\}$ and $c_{\alpha\alpha} \equiv 1$) the properties (4), (5) can be established easily because these channels leave invariant the canonical base in $\mathcal{H}_{\tilde{d}}$, hence $\nu_p(\Phi) = 1, \check{H}(\Phi) = 0$ for such channels. Let us prove for example (4). (Results for a more general class involving channels of such kind are given in [5]).

Let Φ_2 be an arbitrary CP map, and Φ_1 a channel such that $\nu_p(\Phi_1) = 1$. We have

$$\nu_p(\Phi_1 \otimes \Phi_2) = \nu_p((\mathrm{Id}_1 \otimes \Phi_2) \circ (\Phi_1 \otimes \mathrm{Id}_2)) \leq \nu_p(\mathrm{Id}_1 \otimes \Phi_2),$$

where Id denotes the identity channel. Applying the equality $\nu_p\,(\mathrm{Id} \otimes \Phi) = \nu_p\,(\Phi)$ established in [1], we get

$$\nu_p(\Phi_1 \otimes \Phi_2) \leq \nu_p(\Phi_2) = \nu_p(\Phi_1)\nu_p(\Phi_2),$$

whence the multiplicativity follows.

However the proof of multiplicativity for diagonal CP maps, that are not necessarily channels, given in [11], is substantially more complicated (it uses the same method as for the entanglement-breaking maps). Moreover, this proof seems not to be extendable to the more general class of CP maps (14) where $\{\psi_\alpha\}$ is not an orthonormal base, but an arbitrary system of vectors. On the other hand, theorem 1 implies all the multiplicativity/additivity properties for this more general class simply by their complementarity to entanglement-breaking maps and a reference to results in [16, 12]. Specifically, it implies the superadditivity property (6), which so far was known only for direct convex sums of the identity and entanglement-breaking channels (e.g. erasure channel), see [8]. More precisely, theorem 1 combined with proposition 3 from [8] implies property (6) for convex mixtures of either identity or its complementary – completely depolarizing channel – with either entanglement-breaking channel or its complementary. Therefore additivity of (constrained) $\chi-$capacity holds as well for such convex mixtures.

4. Let G be a group and $g \to U_g^A, U_g^B; g \in G; j = 1, 2,$ be two (projective) unitary representations of G in $\mathcal{H}_A$, $\mathcal{H}_B$. The CP map $\Phi : \mathfrak{M}(\mathcal{H}_A) \to \mathfrak{M}(\mathcal{H}_B)$ is *covariant* if

$$\Phi[U_g^A \rho U_g^{A*}] = U_g^B \Phi[\rho] U_g^{B*} \tag{16}$$

for all $g \in G$ and all ρ. The structure of covariant CP maps was studied in the context of covariant dynamical semigroups, see e. g. [6] . In particular, for arbitrary covariant CP map there is the Kraus representation (10), where V_j are the components of a tensor operator for the group G, i. e. satisfy the equations

$$U_g^B V_j U_g^{A*} = \sum_k d_{jk}(g) V_k,$$

where $g \to D(g) = [d_{jk}(g)]$ is a matrix unitary representation of G. It follows that the map complementary to covariant CP map is again covariant, with $D(g)$ playing the role of the second unitary representation.

Let us consider in some detail the extreme transpose-depolarizing channel

$$\Phi(\rho) = \frac{1}{d-1}\left[I\mathrm{Tr}\rho - \rho^T\right],$$

where ρ^T is transpose of ρ in an orthonormal basis $\{e_j\}$ in $\mathcal{H} = \mathcal{H}_A = \mathcal{H}_B, \dim\, \mathcal{H} = d$. This channel breaks the multiplicativity (4) with $\Phi_1 = \Phi_2 = \Phi$ for $d > 3$ and large enough p [18]. At the same time it fulfills the multiplicativity for $1 \leq p \leq 2$ [2] and the additivity (5), see [14], [3]. It has the covariance property

$$\Phi(U\rho U^*) = \bar{U}\Phi(\rho)\bar{U}^*$$

for arbitrary unitary U. Since

$$\Phi(\rho) = \frac{1}{2(d-1)}\sum_{j,k=1}^{d}\left(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|\right)\rho\left(|e_k\rangle\langle e_j| - |e_j\rangle\langle e_k|\right), \tag{17}$$

introducing the index $\alpha = (j,k)$, we have the Kraus representation (10) with operators

$$V_\alpha = \frac{1}{\sqrt{2(d-1)}}\left(|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|\right).$$

Hence

$$\begin{aligned}\tilde{\Phi}(\rho) &= \left[\mathrm{Tr}V_\alpha\rho V_\beta^*\right]_{\alpha,\beta=\overline{1,d}} \\ &= \frac{1}{2(d-1)}\left[\delta_{jj'}\langle e_k|\rho|e_{k'}\rangle - \delta_{jk'}\langle e_k|\rho|e_{j'}\rangle - \delta_{kj'}\langle e_j|\rho|e_{k'}\rangle + \delta_{kk'}\langle e_j|\rho|e_{j'}\rangle\right].\end{aligned}$$

The space $\mathcal{H}_{12}$ in which this matrix acts is tensor product of two $d-$dimensional coordinate spaces with vectors indexed by $k(k')$ and $j(j')$. Let F be the operator in $\mathcal{H}_{12}$ which flips the indices j and k. The expression above takes the form

$$\tilde{\Phi}(\rho) = \frac{1}{2(d-1)}(I_{12} - F)(\rho \otimes I_2)(I_{12} - F). \tag{18}$$

This is the complementary channel which shares the multiplicativity/additivity properties with the channel (17).

By using the decomposition $I_2 = \sum_{j=1}^{d} |e_j\rangle\langle e_j|$, we have the Kraus representation (12) for the complementary channel, where

$$\begin{aligned} \tilde{V}_j|\psi\rangle &= \frac{1}{\sqrt{2(d-1)}}(I_{12} - F)(|\psi\rangle \otimes |e_j\rangle) \\ &= \frac{1}{\sqrt{2(d-1)}}(|\psi\rangle \otimes |e_j\rangle - |e_j\rangle \otimes |\psi\rangle). \end{aligned}$$

The covariance property of the channel (18) is

$$\tilde{\Phi}(U\rho U^*) = (U \otimes U)\tilde{\Phi}(\rho)(U^* \otimes U^*),$$

as follows from the fact that $F(U \otimes U) = (U \otimes U)F$.

The case of depolarizing channel

$$\Phi(\rho) = (1-p)\rho + \frac{p}{d} I \mathrm{Tr}\rho, \quad 0 \le p \le \frac{d^2}{d^2 - 1},$$

can be considered along similar lines[1]. We give only the final result

$$\tilde{\Phi}(\rho) = S(\rho \otimes I_2)S,$$

where

$$S = \sqrt{\frac{p}{d}} I_{12} + \sqrt{d}\left[-\frac{\sqrt{p}}{d} + \sqrt{1 - p\left(\frac{d^2-1}{d^2}\right)}\, |\Omega_{12}\rangle\langle\Omega_{12}| \right],$$

with $|\Omega_{12}\rangle$ the maximally entangled vector in $\mathcal{H} \otimes \mathcal{H}$.

While the depolarizing channel is globally unitarily covariant, the complementary channel has the covariance property

$$\tilde{\Phi}[U\rho U^*] = (U \otimes \bar{U})\tilde{\Phi}[\rho](U \otimes \bar{U})^*$$

for arbitrary unitary operator U in $\mathcal{H}$.

Notice that in both cases the complementary channels have the form

$$\Phi_C(\rho) = S(\rho \otimes I_B)S^*,$$

where $S : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C$ is such that $\mathrm{Tr}_{\mathcal{H}_B} S^* S = I_A$. There is a simple general relation between this representation and the second formula in (1) for

[1] This case was elaborated jointly with N. Datta.

arbitrary CP map. Namely, given $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$ choose an orthonormal basis $\{e'_j\}$ in $\mathcal{H}_B$ and define $S : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C$ by the relation $\langle e'_j|V = S|e'_j\rangle$, or, more precisely,

$$\langle \bar{\psi}_B \otimes \psi_C|V|\psi_A\rangle = \langle \psi_C|S|\psi_A \otimes \psi_B\rangle,$$

where $\bar{\psi}_B$ is complex conjugate in the basis $\{e_j\}$. By interchanging the roles of $\mathcal{H}_B$, $\mathcal{H}_C$ we of course obtain a similar representation for the initial map Φ_B. This is in fact nothing but the dual form (21) of the Stinespring representation, if Φ_B, Φ_C are considered as maps in Heisenberg rather than in Schrödinger picture.

The next important class is Bosonic Gaussian channels [9]. Any such channel can be described as resulting from a quadratic interaction with Gaussian environment. It follows that complementary channel is again Gaussian (see [9], Sec. IVB, for an explicit description). As an example consider attenuation channel with coefficient $k < 1$ described by the transformation

$$a' = ka + \sqrt{1-k^2}a_0$$

in the Heisenberg picture (to simplify notations we write a instead of $a \otimes I_0$ and a_0 instead of $I \otimes a_0$), where the mode a_0 is in a Gaussian state. Complementing this transformation with

$$a'_0 = \sqrt{1-k^2}a - ka_0,$$

we get a canonical (Bogoljubov) transformation implementable by a Hamiltonian quadratic in $a, a_0, a^\dagger, a_0^\dagger$. It follows that the complementary channel is again attenuation channel with the coefficient $\sqrt{1-k^2}$. In the same way, the linear amplifier with coefficient $k > 1$ described by the transformation

$$a' = ka + \sqrt{k^2-1}a_0^\dagger,$$

complements to

$$a'_0 = \sqrt{k^2-1}a^\dagger + ka_0.$$

More detail on complementary covariant and Gaussian channels will be given in a subsequent work.

Note added in replacement: Similar ideas, in the context of channels, are independently developed in the work of C. King, K. Matsumoto, M. Natanson and M. B. Ruskai [13].

Appendix

Theorem 2 *For a CP map $\Phi_B : \mathfrak{M}(\mathcal{H}_A) \to \mathfrak{M}(\mathcal{H}_B)$, there exist a Hilbert space $\mathcal{H}_C$ of dimensionality $d_C \leq d_A d_B$ and an operator $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$, such that the first relation in (1) holds. For any other such operator $V' : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_{C'}$ there is a partial isometry $W : \mathcal{H}_C \to \mathcal{H}_{C'}$ such that*

$$V' = (I_B \otimes W)V, \quad V = (I_B \otimes W^*)V'. \tag{19}$$

Proof. Consider the algebraic tensor product $\mathcal{L} = \mathcal{H}_A \otimes \mathfrak{M}(\mathcal{H}_B)$ generated by the elements $\psi \otimes X,\ \psi \in \mathcal{H}_A,\ X \in \mathfrak{M}(\mathcal{H}_B)$. Let us introduce pre-inner product in $\mathcal{L}$ with the corresponding square of norm

$$\| \sum_j \psi_j \otimes X_j \|^2 = \sum_{j,k} \langle \psi_j || \Phi^*(X_j^* X_k) | \psi_k \rangle = \mathrm{Tr} \sum_{j,k} X_k \Phi(|\psi_k\rangle\langle\psi_j|) X_j^*,$$

where Φ^* is the dual map. This quantity is nonnegative for CP map Φ. After factorizing with respect to the subspace $\mathcal{L}_0$ of zero norm, we obtain the Hilbert space $\mathcal{K} = \mathcal{L}/\mathcal{L}_0$. By construction, $\dim \mathcal{K} \leq d_A d_B^2$.

Put $V\psi = \psi \otimes I$, and $\pi(Y)\Psi = \pi(Y)(\psi \otimes X) = \psi \otimes YX$. Then π is a *-homomorphism $\mathfrak{M}(\mathcal{H}_B) \to \mathfrak{M}(\mathcal{K})$, i. e. a linear map preserving the algebraic operations and the involution: $\pi(XY) = \pi(X)\pi(Y), \pi(X^*) = \pi(X)$. Moreover,

$$\langle \varphi | \Phi^*(X) | \psi \rangle = \langle \varphi \otimes I | \psi \otimes X \rangle = \langle \varphi | V^* \pi(X) V | \psi \rangle, \qquad X \in \mathfrak{M}(\mathcal{H}_B). \tag{20}$$

However any *-homomorphism of the algebra $\mathfrak{M}(\mathcal{H})$ is unitary equivalent to the ampliation $\pi(X) = X \otimes I_C$, where I_C is the unit operator in a Hilbert space $\mathcal{H}_C$, i.e. we can take $\mathcal{K} = \mathcal{H}_B \otimes \mathcal{H}_C$, and (20) takes the form

$$\langle \varphi | \Phi^*(X) | \psi \rangle = \langle \varphi | V^* \left(X \otimes I_C \right) V | \psi \rangle, \qquad X \in \mathfrak{M}(\mathcal{H}_B),$$

or

$$\Phi^*(X) = V^* \left(X \otimes I_C \right) V, \tag{21}$$

which is equivalent to the first equation in (1) with $\Phi_B = \Phi$. It also follows that $\dim \mathcal{H}_C \leq d_A d_B$.

To prove the second statement, consider the subspace

$$\mathcal{M} = \{ (X \otimes I_C) V \psi : \psi \in \mathcal{H}_A, X \in \ \mathfrak{M}(\mathcal{H}_B) \} \subset \mathcal{K} = \mathcal{H}_B \otimes \ \mathcal{H}_C. \tag{22}$$

It is invariant under multiplication by operators of the form $Y \otimes I_C$, hence it has the form $\mathcal{M} = \mathcal{H}_B \otimes \ \mathcal{M}_C, \mathcal{M}_C \subset \mathcal{H}_C$. For a minimal representation we should have $\mathcal{M}_C = \mathcal{H}_C$, because otherwise there would be a proper subrepresentation.

Consider a similar subspace $\mathcal{M}' = \mathcal{H}_{B'} \otimes \mathcal{M}_{C'}$ of the space $\mathcal{K}' = \mathcal{H}_B \otimes \mathcal{H}_{C'}$ for the second dilation. Define the operator R from $\mathcal{M}$ to $\mathcal{M}'$ by

$$R(X \otimes I_C)V\psi = (X \otimes I_{C'})V'\psi. \tag{23}$$

Then R is isometric, since the norms of the vector and of its image under R are both equal to $\langle\psi|\Phi^*(X^*X)|\psi\rangle$ by (20). From (23) we obtain for all $Y \in \mathfrak{M}(\mathcal{H}_B)$

$$R(YX \otimes I_C)V\psi = (Y \otimes I_{C'})R(X \otimes I_C)V'\psi$$

and hence

$$R(Y \otimes I_C) = (Y \otimes I_{C'})R \tag{24}$$

on $\mathcal{M}$. Extend R to the whole of $\mathcal{K}$ by letting it equal to zero on the orthogonal complement to $\mathcal{M}$, then (24) holds on $\mathcal{K}$. Therefore $R = I_C \otimes W$, where W isometrically maps $\mathcal{M}_C$ onto $\mathcal{M}_{C'}$. Relation (23) implies (19). $\square$

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References

[1] G. G. Amosov, A. S. Holevo, and R. F. Werner, On some additivity problems in quantum information theory, Probl. Inform. Transm. **36**, N4, 25 (2000); math-ph/0003002 .

[2] N. Datta, Multiplicativity of maximal p–norms in Werner-Holevo channels for $1 \leq p \leq 2$, quant-ph/0410063.

[3] N. Datta, A. S. Holevo, Y. M. Suhov, Additivity for transpose depolarizing channels, quant-ph/0412034.

[4] I. Devetak, P. Shor, The capacity of a quantum channel for simultaneous transition of classical and quantum information, quant-ph/0311131.

[5] M. Fukuda, Some new additivity results on quantum channels, quant-ph/0505022.

[6] A. S. Holevo, A note on covariant dynamical semigroups, Rep. Math. Phys., **32**, 211 (1993).

[7] A. S. Holevo, On states, channels and purification, Quantum Information Processing **1**, N1, (2002); quant-ph/0204077.

[8] A. S. Holevo, M. E Shirokov, On Shor's channel extension and constrained channels, Commun. Math. Phys. **249**, 417-430, (2004); quant-ph/0306196.

[9] A. S. Holevo, R. F. Werner, Evaluating capacities of Bosonic Gaussian channels. Phys. Rev. A. **63**, 032312, (2001); quant-ph/9912067.

[10] M. Horodecki, P.W. Shor, M.B. Ruskai, General entanglement breaking channels, Rev. Math. Phys. **15**, 629-641, (2003); quant-ph/0302031.

[11] C. King, An application of the matrix inequality in quantum information theory, quant-ph/0412046.

[12] C. King, Maximal p-norms of entanglement breaking channels, quant-ph/0212057.

[13] C. King, K. Matsumoto, M. Natanson and M. B. Ruskai, Properties of conjugate channels with applications to additivity and multiplicativity, quant-ph/0509126.

[14] K. Matsumoto, F. Yura, Entanglement cost of antisymmetric states and additivity of capacity of some channels, quant-ph/0306009.

[15] M. A. Nielsen, I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press 2000.

[16] P. W. Shor, Additivity of the classical capacity of entanglement breaking channels, quant-ph/0201149.

[17] P. W. Shor, Equivalence of additivity questions in quantum information theory, Commun. Math. Phys. **246**, 453-472 (2004); quant-ph/0305035.

[18] R.F.Werner and A.S.Holevo, Counterexample to an additivity conjecture for output purity of quantum channels, J. Math. Phys., **43**, 4353-4357, (2002); quant-ph/0203003.